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Berezinskiĭ–Kosterlitz–Thouless Order in Two-Dimensional O(2)-Ferrofluid

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We study a two-dimensional ferrofluid of hard-core particles with internal degrees of freedom (plane rotators) and $O(2)$-invariant ferromagnetic spin interaction. By reducing the continuous system to an approximating reference lattice system, a lower bound for the two-spin correlation function is obtained. This bound, together with the Fröhlich–Spencer result about the Berezinskiĭ–Kosterlitz–Thouless transition in the two-dimension lattice system of plane rotators, shows that our model also exhibits the same kind of ordering. Namely for a short-range ferromagnetic interaction the two-spin correlation function does not decay faster than some power of the inverse distance between particles, for small temperatures and high densities of the ferrofluid. For a long-range ferromagnetic interaction the model manifests a non-zero order parameter (magnetization) in this domain, whereas for high temperatures spin correlations decay exponentially.

KEY WORDS: Continuous systems; ferrofluids; plane rotators; Berezinskiĭ–Kosterlitz–Thouless transition.

1. INTRODUCTION

This note is motivated by recent[(1–3)] (and not very recent[(4)]) interest to phase transitions in continuous spin systems. There are many rigorous results on phase transition of lattice systems in classical and quantum statistical mechanics. However, we have still only few models of continuum

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systems where the existence of phase transition has been proved. For continuum system of several kind of particles such as the Widom–Rowlinson model,[5] methods using Peierles' argument,[6, 7], Pirogov–Sinai theory,[8] and random cluster expansion[3] are successful. They prove the existence of *spatial* phase transitions separating particles of different type.

Another type of arguments were invented by Gruber and Griffiths (G-G)[4] to study an *orientational* phase transition in a continuum system of particles with internal degrees of freedom, namely "charged" by Ising spins. Hamiltonian of this *Ising ferrofluid* involves two types of interaction: a *direct positional* two-particle interaction with *hard core*, and *ferromagnetic interaction* between spins of those particles. In ref. 4, G-G used a combination of GKS and FKG inequalities to find a lower bound to the ferrofluid magnetization (*orientational* order) in terms of magnetization of an auxiliary lattice Ising model. In ref. 1, the G-G model was generalized to a continuum system of particles equipped with essentially one dimensional continuous spin. The existence of the *orientational* order for low temperatures and high densities was proved for this system using the G-G arguments combined with Wells' inequality.[9, 10]

The purpose of the present paper is to apply the G-G method to prove the existence of the Berezinskiĭ–Kosterlitz–Thouless *orientational* ordering in $O(2)$-symmetric spin ferrofluid. We consider a *two-dimensional* continuum model of particles carrying two-component unit vector spins (*plane rotators*). As in refs. 1 and 4 the interaction constitutes of two parts: *positional* two-particle interaction with a hard-core, and *ferromagnetic* $O(2)$-invariant two-spin interaction between those particles. It is known[11, 12] that for a *short-range* spin–spin interaction there is no spontaneous breaking of the $O(2)$ symmetry in this model: the Gibbs state is $O(2)$-invariant for all densities and any non-zero temperature. This implies that the magnetic order parameter is always null manifesting the Mermin–Wagner theorem known for lattice $O(n)$-symmetric spin systems in one and two dimensions, see, e.g., ref. 13. The main result of the present paper is the following:

Consider the two-spin correlation function of the model. We prove that it can be bounded from below by that of an *auxiliary lattice* spin system. Combining this estimate with the Fröhlich–Spencer result,[14] we find that for low temperatures and high particle densities (chemical potentials) this two-spin correlation function does not decay faster than some power of the inverse distance between particles. This means that the two-dimensional $O(2)$-symmetric spin ferrofluid with a *short-range* ferromagnetic interaction manifests the Berezinskiĭ–Kosterlitz–Thouless phase transition.[15, 16] On the other hand, for a *long-range* ferromagnetic interaction the model shows a non-zero magnetic order parameter in this domain of temperatures and chemical potentials. Since the particle density is bounded from above by the closest packing, the standard arguments[13] imply exponential decay of the spin–spin correlations for high temperatures and any chemical potentials.

Notice that as in refs. 1 and 4, we study here only a problem of *orientational* order in the spin subsystem. The question of correlation between *orientational* and *spatial* orders is much more difficult even for lattice systems.[17–20] For correlations between *orientational* and *spatial* orders in mean-field models of ferrofluids see, e.g., ref. 2. We would like to mention also a recent progress concerning *spatial* order in *spinless* liquids.[21]

The paper is organized as follows: We give an explicit description of the model in Section 2. In Section 3, we construct an auxiliary reference lattice model to compare the spin–spin correlation functions in this model and in the original continuum model. There, we apply a version of the G-G approach, which in our case is based on Ginibre, instead of GKS, inequalities.[22] Finally (Section 4) the Wells inequality[9] allows us to find a lower bound of the two-spin correlation function in the auxiliary lattice system via the correlation function of the lattice two-dimensional plane-rotator model. This, together with the Fröhlich–Spencer result,[14] gives our main result. In conclusion (Section 5) we make some remarks concerning the expected phase diagram of our model.

2. THE MODEL

We consider a two-dimensional classical continuous system of identical particles with internal degrees of freedom $\phi \in S^1$ (*plane rotator*) moving in a bounded domain $\Lambda \subset \mathbb{R}^2$. A configuration of the system with n particles is denoted by

$$(X_n, \Phi_n) := (x_1, \phi_1; ...; x_n, \phi_n) \in (\Lambda \times S^1)^n, \tag{2.1}$$

where $\phi_j = (\cos\theta_j, \sin\theta_j) \in S^1$. We write simply (X, Φ), with $n = |X|$ to be the number of particles, and we shall use the natural notations:

$$(X, \Phi) \cup (X', \Phi') = (x_1, \phi_1; ...; x_n, \phi_n; x'_1, \phi'_1; ...; x'_{n'}, \phi'_{n'}), \tag{2.2}$$

and

$$(x, y, X, \phi_x, \phi_y, \Phi) = (x, \phi_x; y, \phi_y; x_1, \phi_1; ...; x_n, \phi_n). \tag{2.3}$$

We assume that the particles interact via *two kinds* of two-body potentials, i.e., the Hamiltonian of the system is composed of two parts:

$$H_\Lambda(X, \Phi) = U(X) + V(X, \Phi), \tag{2.4}$$

where the first term represents a two-body spin-independent interaction:

$$U(X) = \sum_{\{x_i, x_j\} \subset X} u(|x_i - x_j|), \tag{2.5}$$

whereas the second term corresponds to the spin–spin interaction:

$$V(X, \Phi) = - \sum_{\{x_i, x_j\} \subset X} J(|x_i - x_j|)(\phi_i, \phi_j), \tag{2.6}$$

with the scalar product $(\phi_i, \phi_j) = \cos(\theta_i - \theta_j)$. We put $H_\Lambda(X, \Phi) = 0$ for $|X| = 0, 1$.

We assume that the two-body potential u and the spin–spin interaction J satisfy the following conditions:

(u1) $u(t) = \infty$ for $t < 2R$, i.e., each particle has a *hard-core* of radius R;

(u2) u is a bounded measurable function on $[2R, +\infty)$, and it decreases as $t^{-(2+\varepsilon)}$,, $\varepsilon > 0$, for $t \to \infty$;

(J1) $J(t) \geqslant 0$ for $t \in [0, \infty)$ (*ferromagnetic* interaction) such that

$$J^0 := \inf_{0 \leqslant t \leqslant (4R+\delta)\, 5^{1/2}} J(t) > 0 \tag{2.7}$$

for some $\delta > 0$;

(J2) the spin–spin interaction is *regular* in the sense that

$$\|J\| = \sup_{n \geqslant 2, \Lambda} \; \sup_{X_n \in \Lambda^n_{\text{adm}}} \; \sum_{x_j \in X_n \setminus \{x_1\}} J(|x_1 - x_j|) < \infty, \tag{2.8}$$

where $\Lambda^n_{\text{adm}} := \{X_n = (x_1,..., x_n) \in \Lambda^n : |x_j - x_k| \geqslant 2R\}$ denotes the set of *admissible* hard-core particle configurations. A simple example of regular $J(t)$ is given by a finite-range continuous interaction.

These conditions guarantee the convergence of the grand-canonical partition function:

$$\Xi_\Lambda(\beta, \mu) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Lambda \times S^1)^n} dX_n \, d\Phi_n \, e^{\beta\mu n - \beta H_\Lambda(X_n, \Phi_n)} \tag{2.9}$$

for the inverse temperature $\theta^{-1} = \beta > 0$ and chemical potential $\mu \in \mathbb{R}$, as well as existence of the grand-canonical Gibbs measure on Λ

$$\langle - \rangle_{\Lambda, \beta, \mu} = \Xi_\Lambda(\beta, \mu)^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Lambda \times S^1)^n} dX_n \, d\Phi_n (-) \, e^{\beta\mu n - \beta H_\Lambda(X_n, \Phi_n)} \tag{2.10}$$

and its infinite-volume limit Gibbs measure, see, e.g., ref. 13. Here $dX_n = dx_1 \cdots dx_n$ stands for the Lebesgue measure on $\mathbb{R}^{2n}$, and $d\Phi_n = d\phi_1 \cdots d\phi_n$, where $d\phi_j = d\theta_j/2\pi$ is the normalized invariant (Haar) measure on the unit circle S^1, and by definition we put $H_\Lambda(X_n, \Phi_n) = 0$ for $n = 0, 1$.

Notice that the properties $(u1)$, $(u2)$ imply the existence of the constant $A_+ \in \mathbb{R}$ such that

$$\sup_{n \geqslant 2,\, \Lambda} \ \sup_{X_n \in \Lambda^n_{\mathrm{adm}}} \ \sum_{x_j \in X_n \setminus \{x_1\}} u(|x_1 - x_j|) \leqslant A_+. \tag{2.11}$$

Hereafter we make use the following notations. Denote by $G = \{G_n\}_{n \geqslant 0}$ a sequence of real measurable *symmetric* functions $G_n = G_n(X_n, \Phi_n)$ on $(\mathbb{R}^1 \times S^1)^n$, for $n \geqslant 1$, with $G_0 \in \mathbb{R}^1$.

For any measurable $K \subset \mathbb{R}^2$, we introduce the set $\mathscr{P}(K) := \bigcup_{n=0}^{\infty} K^n = \{\varnothing\} \cup \mathscr{P}'(K)$, and we define:

$$\int_{\mathscr{P}(K)} dX\, d\Phi\, G(X, \Phi) := G_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(K \times S^1)^n} dX_n\, d\Phi_n\, G_n(X_n, \Phi_n)$$

$$= G_0 + \int_{\mathscr{P}'(K)} dX\, d\Phi\, G(X, \Phi).$$

Let $\{K_l\}_{l \in L}$ be a finite family of measurable non-intersecting subsets of $\mathbb{R}^2$. Then one gets the following identity:

$$\int_{\mathscr{P}(\bigcup_{l \in L} K_l)} dX\, d\Phi\, G(X, \Phi)$$

$$= G_0 + \sum_{M \subseteq L:\, M \neq \varnothing} \ \prod_{l \in M} \int_{\mathscr{P}'(K_l)} dX^{(l)}\, d\Phi^{(l)}\, G(X_M, \Phi_M), \tag{2.12}$$

where $X_M = \bigcup_{l \in M} X^{(l)}$ and $\Phi_M = \bigcup_{l \in M} \Phi^{(l)}$. Indeed, one gets formula (2.12) by induction in the cardinality $|L|$ of the set L. For example, let $L = \{1, 2\}$, i.e., we have $\{K_1, K_2\}$ with $K_1 \cap K_2 = \{\varnothing\}$. Then

$$\int_{\mathscr{P}(K_1 \cup K_2)} dX\, d\Phi\, G(X, \Phi)$$

$$= G_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(K_1 \cup K_2)^n} dX_n \int_{(S^1)^n} d\Phi_n\, G_n(X_n, \Phi_n)$$

$$= G_0 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} \int_{(K_1)^m} dY_m \int_{(K_2)^{n-m}} dZ_{n-m}$$

$$\times \int_{(S^1)^n} d\Phi_n\, G_n(Y_m \cup Z_{n-m}, \Phi_n)$$

$$= G_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(K_1)^n} dY_n \int_{(S^1)^n} d\Phi_n \, G_n(Y_n, \Phi_n)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(K_2)^n} dZ_n \int_{(S^1)^n} d\Phi_n \, G_n(Z_n, \Phi_n)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m!\, n!} \int_{(K_1)^m} dY_m \int_{(K_2)^n} dZ_n$$

$$\times \int_{(S^1)^{m+n}} d\Phi_{m+n} \, G_{m+n}(Y_m \cup Z_n, \Phi_{m+n})$$

$$= G_0 + \int_{\mathscr{P}'(K_1)} dY \, d\Phi \, G(Y, \Phi) + \int_{\mathscr{P}'(K_2)} dZ \, d\Phi \, G(Z, \Phi)$$

$$+ \int_{\mathscr{P}'(K_1)} dY \, d\Phi \int_{\mathscr{P}'(K_2)} dZ \, d\Phi \, G(Y \cup Z, \Phi \cup \Phi'),$$

which holds because of symmetry of the functions $\{G_n\}_{n \geqslant 0}$.

Let $x \neq y$, and let the observable $F_{x,y} = \{(F_{x,y})_n\}_{n \geqslant 0}$ be defined for $n \geqslant 2$ by

$$(F_{x,y})_n \, (X_n, \Phi_n) := \sum_{\{x_i, x_j\} \subset X_n} (\phi_i, \phi_j) \, \delta(x - x_i) \, \delta(y - x_j), \tag{2.13}$$

and by $(F_{x,y})_n = 0$, for $n = 0, 1$. Then the *two-spin correlation function* in the grand-canonical ensemble is the *expectation* of the functions (2.13) with respect to the Gibbs measure (2.10):

$$\langle F_{x,y} \rangle_{\Lambda, \beta, \mu} = \Xi_\Lambda(\beta, \mu)^{-1} \int_{\mathscr{P}(\Lambda)} dX \, d\Phi \, e^{\beta\mu |X|} e^{-\beta H_\Lambda(X, \Phi)} F_{x,y}(X, \Phi)$$

$$= \Xi_\Lambda(\beta, \mu)^{-1} \int_{\mathscr{P}(\Lambda)} dX \, d\Phi \int_{S^1} d\phi_x$$

$$\times \int_{S^1} d\phi_y (\phi_x, \phi_y) \, e^{\beta\mu(|X|+2)} e^{-\beta H_\Lambda(x, y, X, \phi_x, \phi_y, \Phi)}. \tag{2.14}$$

3. COMPARISON WITH A LATTICE MODEL

Let $\mathbb{Z}^2_a$ be a square lattice with side $a = 4R + \delta$ for some $\delta > 0$, cf. (2.7). Let $\square_\ell = ((\ell_1 - \frac{1}{2}a), (\ell_1 + \frac{1}{2}a)] \times ((\ell_2 - \frac{1}{2}a), (\ell_2 + \frac{1}{2}a)]$ denote the semi-open plaquette with side a centered at site $\ell \in \mathbb{Z}^2{}_a$. Below we consider connected domain $\Lambda \subset \mathbb{R}^2$ which is a union of these non-intersecting plaquettes:

$$\Lambda = \bigcup_{\ell \in L} \Box_\ell, \tag{3.1}$$

for a finite subset $L \subset \mathbb{Z}^2{}_a$.

The aim of this section is to construct on the sublattice $L \subset \mathbb{Z}^2{}_a$ an *auxiliary lattice-gas* $O(2)$-spin model, which gives an estimate of the correlation function (2.14) from below, see Theorem 1. To this end we associate with each plaquette $\ell \in \mathbb{Z}^2{}_a$ two random variables: n_ℓ (*lattice-gas occupation number*) and ϕ_ℓ (*spin orientation*) taking their values in $\{0, 1\}$ and in S^1 respectively. Adapting the arguments of refs. 1 and 4 we shall prove that there is a domain of (β, μ) such that

$$\langle F_{x,y} \rangle_{\Lambda, \beta, \mu} \geqslant C \langle n_{\ell_x} n_{\ell_y} (\phi_{\ell_x}, \phi_{\ell_y}) \rangle_{L, \beta, \mu_0}, \tag{3.2}$$

for some $C > 0$, $\mu_0 \in \mathbb{R}$ independent of x, y and Λ. Here the sites ℓ_x and ℓ_y are defined by the condition: $\ell_x = \{\ell \in \mathbb{Z}^2{}_a : x \in \Box_\ell\}$ and $\ell_y = \{\ell \in \mathbb{Z}^2{}_a : y \in \Box_\ell\}$. In the following we assume that $x \neq y$ are such that the corresponding plaquettes $\Box_{\ell_x}$, $\Box_{\ell_y}$ are distinct. The expectation value in the right-hand side of (3.2) is over the Gibbs measure for the *lattice-gas* ferromagnetic $O(2)$-spin Hamiltonian:

$$H^0_L(n_L, \phi_L) := -\tfrac{1}{2} \sum_{\{\ell, \ell'\} \subset L} J^0_{\ell\ell'} n_\ell n_{\ell'} (\phi_\ell, \phi_{\ell'}) - \mu_0 \sum_{\ell \in L} n_\ell \tag{3.3}$$

with $J^0_{\ell\ell'} \geqslant 0$ and the chemical potential μ_0.

We present our construction of this auxiliary model via the sequence of the following

Remarks. (a) Since by (3.1) the domain Λ is the union of non-intersecting plaquettes, one can use (2.12) to rewrite the two-spin correlation function (2.14) in the form:

$$\langle F_{x,y} \rangle_{\Lambda, \beta, \mu} = \Xi_\Lambda(\beta, \mu)^{-1} \Bigg[\int_{S^1} d\phi_x \int_{S^1} d\phi_y (\phi_x, \phi_y)\, e^{2\beta\mu} e^{-\beta H_\Lambda(x, y, \phi_x, \phi_y)}$$
$$+ \sum_{M \subseteq L : M \neq \varnothing} \prod_{\ell \in M} \int_{\mathscr{P}'(\Box_\ell)} dX^{(\ell)}\, d\Phi^{(\ell)}\, e^{\beta\mu |X^{(\ell)}|}$$
$$\times \int_{S^1} d\phi_x \int_{S^1} d\phi_y (\phi_x, \phi_y)\, e^{2\beta\mu} e^{-\beta H_\Lambda(x, y, X_M, \phi_x, \phi_y, \Phi_M)} \Bigg]$$

This yields the representation:

$$\langle F_{x,y}\rangle_{\Lambda,\beta,\mu} = \Xi_\Lambda(\beta,\mu)^{-1} \sum_{M\subseteq L} \prod_{\ell\in M} \int_{\mathscr{P}'(\square_\ell)} dX^{(\ell)}\, d\Phi^{(\ell)}\, e^{\beta\mu|X^{(\ell)}|} \times \{e^{-\beta H_\Lambda(X_M,\Phi_M)} \langle(\phi_x,\phi_y)\,|\,X_M\rangle\, R_{x,y}(X_M)\}, \tag{3.4}$$

where we put

$$\langle(\phi_x,\phi_y)\,|\,X_M\rangle := \frac{\int_{S^1} d\phi_x \int_{S^1} d\phi_y \int_{S^{|M|}} d\Phi_M (\phi_x,\phi_y)\, e^{-\beta V(x,y,X_M,\phi_x,\phi_y,\Phi_M)}}{\int_{S^1} d\phi_x \int_{S^1} d\phi_y \int_{S^{|M|}} d\Phi_M\, e^{-\beta V(x,y,X_M,\phi_x,\phi_y,\Phi_M)}}, \tag{3.5}$$

and

$$R_{x,y}(X_M) := e^{\beta\{2\mu+U(X_M)-U(x,y,X_M)\}} \frac{\int_{S^1} d\phi_x \int_{S^1} d\phi_y \int_{S^{|M|}} d\Phi_M\, e^{-\beta V(x,y,X_M,\phi_x,\phi_y,\Phi_M)}}{\int_{S^{|M|}} d\Phi_M\, e^{-\beta V(X_M,\Phi_M)}}. \tag{3.6}$$

By estimates (2.8) and (2.11) we get the lower bound for (3.6):

$$R_{x,y}(X_M) \geqslant e^{2\beta\{\mu-A_+-\|J\|\}} := C. \tag{3.7}$$

By definition of (3.5) the right-hand side is the two-spin correlation function of a ferromagnetic *lattice* $O(2)$-spin model on the sites (x, y, X_M). Now with the estimate (3.7) in hand we return to the proof of the lower bound (3.2).

(b) By the Ginibre inequalities for the plane-rotator ferromagnets,[22] the expectation $\langle(\phi_x,\phi_y)\,|\,X_M\rangle$ is an *increasing* function of the set of ferromagnetic bonds $\{J_{z,w}\}_{z,w\in(x,y,X_M)}$. Therefore, we get for it the estimate from below:

$$\langle(\phi_x,\phi_y)\,|\,X_M\rangle \geqslant \langle(\phi_x,\phi_y)\,|\,X_M^{\min}\rangle, \tag{3.8}$$

where the configuration $(x, y, X_M^{\min})$ is obtained from (x, y, X_M) by keeping exactly *one* particle in each of *non-empty* plaquette $\{\ell\in L : \square_\ell \cap (x,y,X_M) \neq \{\varnothing\}\} = \{\ell_x,\ell_y\}\cup M$. One can keep them arbitrary *except* in the plaquettes $\square_{\ell_x}$, $\square_{\ell_y}$, where we must choose them coinciding with particles x, y. Otherwise, the right-hand side in (3.8) will be identically zero.

(c) Using again the Ginibre inequalities we can make the right-hand side of (3.8) lower, if we replace in the interaction V the ferromagnetic couplings by $J^0_{\ell\ell'} = \inf_{z\in\square_\ell, w\in\square_{\ell'}} J(|z-w|)$. The estimate runs in the same direction if we put $J^0_{\ell_x\ell_y} = J^0_{\ell_x\ell} = J^0_{\ell\ell_y} = 0$ for $\ell\in M$ in the cases $\ell_x \notin M$, or

$\ell_y \notin M$. Therefore, the right-hand side of (3.8) can be bounded from below by the two-spin correlation function of a lattice system with sites localized *only* on the non-empty plaquettes $M \subseteq L$:

$$\langle (\phi_x, \phi_y) \mid X_M^{\min} \rangle \geqslant \langle (\phi_{\ell_x}, \phi_{\ell_y}) \rangle_{V_M^0} := \langle F_{x,y}^0 \rangle_{L,\beta} (M), \tag{3.9}$$

where expectation in the right-hand side of (3.9) is calculated with the *lattice-gas* spin interaction:

$$V_M^0(\phi_M) := - \sum_{\{\ell, \ell'\} \subset M} J_{\ell\ell'}^0(\phi_\ell, \phi_{\ell'}). \tag{3.10}$$

Here again we put $V_M^0 = 0$ for $|M| = 0, 1$.

Notice that the integration over $d\phi_x \, d\phi_y$ implies that $\langle F_{x,y}^0 \rangle_{L,\beta} (M) = 0$ unless $(\ell_x, \ell_y) \in M$, or if $|M| = 0, 1$.

(d) Since the lower bound in (3.9) depends only on M, we introduce the grand-canonical partition function for a given configuration M of occupied plaquettes by:

$$\Xi(M) := \prod_{\ell \in M} \left(\int_{\mathscr{P}'(\square_\ell)} dX^{(\ell)} \, d\Phi^{(\ell)} \, e^{\beta\mu |X^\ell|} \right) e^{-\beta H_\Lambda(X_M, \Phi_M)}, \tag{3.11}$$

with the convention $\Xi(\emptyset) = 1$. Then by (2.9), (2.12), and (3.1) one gets

$$\Xi_\Lambda(\beta, \mu) = \sum_{M \subseteq L} \Xi(M). \tag{3.12}$$

Therefore, from (3.4) and (3.11) we can estimate the two-spin correlation function with the help of (3.8) and (3.9) as:

$$\langle F_{x,y} \rangle_{\Lambda,\beta,\mu} \geqslant C \sum_{M \subseteq L} P_{\beta,\mu}(M) \langle F_{x,y}^0 \rangle_{L,\beta} (M), \tag{3.13}$$

where

$$P_{\beta,\mu}(M) := \frac{\Xi(M)}{\Xi_\Lambda(\beta, \mu)} \tag{3.14}$$

is the probability measure induced by $\{\Xi(M)\}_{M \subseteq L}$.

(e) For a given configuration $M \subseteq L$ the spin partition function for interaction (3.10) has the form:

$$Z_M := \left(\prod_{j \in M} \int_{S^1} d\phi_j \right) e^{-\beta V_M^0(\phi_M)},$$

with the usual convention $Z_{M=\varnothing}=1$, and

$$p_{\mu_0}(M) := \frac{Z_M e^{\beta\mu_0 |M|}}{\sum_{M'\subseteq L} Z_{M'} e^{\beta\mu_0 |M'|}} \tag{3.15}$$

defines the *lattice-gas* probability measure on L with chemical potential $\mu_0 \in \mathbb{R}$.

Proposition 1 (Lebowitz inequality). For any finite μ_0 the probability (3.15) verifies the FKG-condition:

$$p_{\mu_0}(M \cup M')\, p_{\mu_0}(M \cap M') \geqslant p_{\mu_0}(M)\, p_{\mu_0}(M'). \tag{3.16}$$

Proof. The proof follows through verbatim along the arguments of ref. 23 with the substitution of the GKS inequality for ferromagnetic systems by the Ginibre inequality for the model (3.10). By definition (3.15) it is enough to prove that

$$F := \ln Z_{M\cup M'} - \ln[Z_M Z_{M'}/Z_{M\cap M'}] \geqslant 0.$$

Let $J^0_{\ell\ell'}=0$ for all $\ell \in M \cap M'$ and $\ell' \in M' \setminus M$. Then the inequality for F is a consequence of the Ginibre inequalities. Since the Ginibre inequalities imply also that

$$\frac{\partial F}{\partial J^0_{\ell\ell'}} = \langle(\phi_\ell, \phi_{\ell'})\rangle_{V^0_{M\cup M'}} - \langle(\phi_\ell, \phi_{\ell'})\rangle_{V^0_{M'}} \geqslant 0, \tag{3.17}$$

$F \geqslant 0$ holds for $J^0_{\ell\ell'} > 0$. ∎

Notice that inequality (3.17) express the monotonicity:

$$\langle F^0_{x,y}\rangle_{L,\beta}\,(M) \leqslant \langle F^0_{x,y}\rangle_{L,\beta}\,(M'), \tag{3.18}$$

for $M \subseteq M'$, which we have already discussed above in (b).

(f) Since $p_{\mu_0}(M) > 0$ we can represent the probability (3.14) as

$$P_{\beta,\mu}(M) := f(M)\, p_{\mu_0}(M). \tag{3.19}$$

Lemma 1. The $f(M)$ is monotonous increasing function of the configuration M, i.e., $f(M) \leqslant f(M')$ for $M \subseteq M'$ whenever (β, μ) satisfies for a given μ_0 the following condition:

$$e^{\beta(A_+ + 2\|J\| - \mu + \mu_0)} \leqslant (a-4R)^2. \tag{3.20}$$

Proof. By definition (3.19) the monotonicity $f(M) \leqslant f(M')$ for $M \subseteq M'$, is equivalent to the inequality

$$\frac{Z_{M'} e^{\beta\mu_0 |M' \setminus M|}}{Z_M} \leqslant \frac{\Xi(M')}{\Xi(M)}. \tag{3.21}$$

Let $M' = M \cup \{j\}$. Then by (3.11) for exactly *one* supplementary particle in the plaquette $\square_j$ one gets the estimate:

$$\begin{aligned}\frac{\Xi(M \cup \{j\})}{\Xi(M)} &\geqslant \Xi_M^{-1} \prod_{\ell \in M} \left(\int_{\mathscr{P}'(\square_\ell)} dX_\ell \int_{S^{|X_\ell|}} d\Phi(X_\ell)\, e^{\beta\mu |X_\ell|} \right) \int_{\square_j} dw \int_{S^1} d\phi_w\, e^{\beta\mu} \\ &\quad \times e^{-\beta\{V(X_M, \Phi_M) + U(X_M)\}} e^{-\beta\{V(w, X_M, \phi_w, \Phi_M) - V(X_M, \Phi_M)\}} \\ &\quad \times e^{-\beta\{U(w, X_M) - U(X_M)\}} \\ &\geqslant e^{-\beta(-\mu + A_+ + \|J\|)} (a - 4R)^2 \end{aligned} \tag{3.22}$$

since at least a domain of the volume $(a-4R)^2$ in the plaquette $\square_j$ is free for the particle w. Similarly we get from (3.10) that

$$\frac{Z_{M \cup \{j\}}}{Z_M} e^{\beta\mu_0} \leqslant e^{\beta(\mu_0 + \|J\|)}. \tag{3.23}$$

The estimates (3.22) and (3.23) give the inequality (3.21) for $M' = M \cup \{j\}$. Iterating in j one gets it for any $M \subseteq M'$. ▮

Theorem 1. Let μ_0 be some fixed value. Then in the domain of temperatures and chemical potentials defined by the condition (3.20) the two-spin correlation function of the $O(2)$-ferrofluid is bounded from below:

$$\langle F_{x,y} \rangle_{\Lambda, \beta, \mu} \geqslant C \sum_{M \subseteq L} p_{\mu_0}(M) \langle F^0_{x,y} \rangle_{L,\beta} (M), \tag{3.24}$$

where C is defined by (3.7).

Proof. Since the probability $p_{\mu_0}(M)$ verifies the FKG condition (3.16) and both functions $f(M)$ and $\langle F^0_{x,y} \rangle_{L,\beta} (M)$ are monotonous increasing with M, the FKG inequality[24] gives:

$$\begin{aligned}&\sum_{M \subseteq L} p_{\mu_0}(M)\, f(M) \langle F^0_{x,y} \rangle_{L,\beta} (M) \\ &\qquad \geqslant \left\{ \sum_{M \subseteq L} p_{\mu_0}(M) \langle F^0_{x,y} \rangle_{L,\beta} (M) \right\} \left\{ \sum_{M \subseteq L} p_{\mu_0}(M)\, f(M) \right\}. \end{aligned} \tag{3.25}$$

Then by the inequality (3.13) and the definition (3.19) one readily obtains the lower bound (3.24). ▮

Remark 1. Let $n_L := \{n_\ell\}_{\ell \in L}$ be the lattice-gas *occupation-number* variables defined by

$$n_\ell(M) = \begin{cases} 1, & \text{if} \quad \ell \in M, \\ 0, & \text{if} \quad \ell \in L \setminus M. \end{cases} \tag{3.26}$$

Then interaction (3.10) takes the form:

$$V^0_M(\phi_M) = -\sum_{\{\ell, \ell'\} \subset L} J^0_{\ell\ell'} n_\ell(M)\, n_{\ell'}(M)(\phi_\ell, \phi_{\ell'}) := V^0_L(n_L(M)\, \phi_L). \tag{3.27}$$

With this notations we can rewrite the right-hand side of (3.24) as:

$$\sum_{N \subset L} p_{\mu_0}(N) \langle F^0_{x,y} \rangle_{L, \beta, N} = (Z_L(\beta, \mu_0))^{-1} \sum_{\{n_\ell = 0, 1 : \ell \in L\}} \int_{(S^1)^{|L|}} \left(\prod_{\ell \in L} e^{\beta \mu_0 n_\ell}\, d\phi_\ell \right) \times n_{\ell_x} n_{\ell_y} (\phi_{\ell_x}, \phi_{\ell_y})\, e^{-\beta V^0_L(n_L, \phi_L)}. \tag{3.28}$$

This proves the result announced at the beginning of this section in (3.2), (3.3) with $H^0_L(n_L, \phi_L) = V^0_L(n_L, \phi_L) - \mu_0 \sum_{\ell \in L} n_\ell$.

The right-hand side of (3.28) is obviously identical to

$$(Z_L(\beta, \mu_0))^{-1} \left(\prod_{\ell \in L} \int_{\mathbb{R}} \{\delta(r_\ell) + e^{\beta\mu_0} \delta(r_\ell - 1)\}\, dr_\ell \int_{S^1} d\phi_\ell \right) \times r_{\ell_x} r_{\ell_y} (\phi_{\ell_x}, \phi_{\ell_y})\, e^{-\beta V^0_L(r_L, \phi_L)}, \tag{3.29}$$

where

$$Z_L(\beta, \mu_0) = \left(\prod_{\ell \in L} \int_{\mathbb{R}} \{\delta(r_\ell) + e^{\beta\mu_0} \delta(r_\ell - 1)\}\, dr_\ell \int_{S^1} d\phi_\ell \right) e^{-\beta V^0_L(r_L, \phi_L)},$$

and $\{d\phi_\ell = d\theta_\ell / 2\pi\}_{\ell \in L}$, for $\theta_\ell \in [0, 2\pi]$, with $(\phi_{\ell_x}, \phi_{\ell_y}) = \cos(\theta_{\ell_x} - \theta_{\ell_y})$. We use the representation (3.29) in the next section.

4. THE WELLS INEQUALITY AND POWER DECAY OF CORRELATIONS

In this section, we apply the Wells inequality[9, 10] to obtain a lower bound of (3.29) via the two-spin correlation function of the standard non-diluted $O(2)$-ferromagnetic model on $\mathbb{Z}^2$. Since we need an *explicit quantitative* estimate of domain of validity of this bound, we derive the Wells inequality for the case of ferromagnetic plane rotators. In fact below we do this in a more general setting than we need for our purposes.

Proposition 2 (Wells inequality for rotators). Let $\epsilon = \min\{1/2, b/(a+b)\}$ with $a, b > 0$. Let the measure $d\nu_{ab}(r) = \{a\delta(r) + b\delta(r-1)\}\, dr$. Then for arbitrary subsets $A \subset L$ one has the following inequality:

$$Z_{\nu_{ab}}^{-1}\left(\prod_{j\in L}\int_{\mathbb{R}} d\nu_{ab}(r_j)\int_0^{2\pi} d\theta_j\right) r^A \cos\theta_A\, e^{\sum_{B\subset L} J_B r^B \cos\theta_B}$$

$$\geqslant Z_\epsilon^{-1}\left(\prod_{j\in L}\int_{\mathbb{R}} \delta(r_j-\epsilon)\, dr_j\int_0^{2\pi} d\theta_j\right) r^A \cos\theta_A\, e^{\sum_{B\subset L} J_B r^B \cos\theta_B}, \tag{4.1}$$

where $J_C \geqslant 0$, $r^C = \prod_{j\in C} r_j$, $\theta_C = \sum_{j\in C} k_C(j)\,\theta_j$ with $k_C\colon C \to \mathbb{Z}$ for any $C \subset L$, and $Z_{\nu_{ab}}$, Z_ϵ are the corresponding normalizing factors.

Proof. We follow essentially the remarks in Appendix of ref. 10. By the Ginibre method of duplicate variables, the inequality (4.1) is equivalent to

$$\left(\prod_{j\in L}\int_{\mathbb{R}} d\nu_{ab}(\rho_j)\int_{\mathbb{R}} \delta(r_j-\epsilon)\, dr_j\int_0^{2\pi} d\theta_j\int_0^{2\pi} d\theta'_j\right)$$
$$\times(\rho^A\cos\theta_A - r^A\cos\theta'_A)\, e^{\sum_{B\subset L} J_B(\rho^B\cos\theta_B + r^B\cos\theta'_B)} \geqslant 0.$$

Since,

$$\rho^A\cos\theta_A - r^A\cos\theta'_A$$
$$= (\rho^A + r^A)\sin\frac{\theta'_A+\theta_A}{2}\sin\frac{\theta'_A-\theta_A}{2} + (\rho^A - r^A)\cos\frac{\theta'_A+\theta_A}{2}\cos\frac{\theta'_A-\theta_A}{2}$$

$$\rho^B\cos\theta_B + r^B\cos\theta'_B$$
$$= (\rho^B + r^B)\cos\frac{\theta'_B+\theta_B}{2}\cos\frac{\theta'_B-\theta_B}{2} + (\rho^B - r^B)\sin\frac{\theta'_B+\theta_B}{2}\sin\frac{\theta'_B-\theta_B}{2},$$

after developing the exponent one gets that the integrand in the left-hand side of the above inequality is a linear combination of terms which have the form:

$$\left(\prod_{C\subset L}(\rho^C + r^C)^{n_C}(\rho^C - r^C)^{m_C}\right) f_{n_C,m_C}\left(\frac{\theta'_C+\theta_C}{2}\right) f_{n_C,m_C}\left(\frac{\theta'_C-\theta_C}{2}\right),$$

where $f_{n_C,m_C}(\varphi)$, as well as the product $f_{n_C,m_C}((\theta'+\theta)/2)\, f_{n_C,m_C}((\theta'-\theta)/2)$, are periodic functions with period 2π with respect to each of variables $\varphi_j, \theta'_j, \theta_j$ $(j \in C)$ for all n_C and m_C.

Now, by standard Ginibre's arguments (see [22, Section 2]) one gets:

$$\left(\prod_{j\in L}\int_0^{2\pi} d\theta_j \int_0^{2\pi} d\theta'_j\right) f_{n_C, m_C}\left(\frac{\theta'_C+\theta_C}{2}\right) f_{n_C, m_C}\left(\frac{\theta'_C-\theta_C}{2}\right) \geqslant 0.$$

Then expanding

$$\rho^C = \prod_{j\in C}\left(\frac{\rho_j+r_j}{2}+\frac{\rho_j-r_j}{2}\right),$$

and

$$r^C = \prod_{j\in C}\left(\frac{\rho_j+r_j}{2}-\frac{\rho_j-r_j}{2}\right),$$

it is enough to show that for all non-negative integers k, l one has:

$$\int_{\mathbb{R}} d\nu_{ab}(\rho)\int_{\mathbb{R}} \delta(r-\epsilon)\, dr(\rho+r)^k\,(\rho-r)^l \geqslant 0,$$

which is equivalent to

$$b(1+\epsilon)^k\,(1-\epsilon)^l+(-1)^l\, a\epsilon^{k+l} \geqslant 0. \tag{4.2}$$

This is true for even l. So let l be odd. Note that function $[\epsilon/(1+\epsilon)]^k$ $[\epsilon/(1-\epsilon)]^l$ is increasing in $\epsilon \in [0, 1)$ and non-increasing in $k \geqslant 0$, $l > 0$ as soon as $\epsilon \in [0, 1/2]$. Therefore, we have for those k, l and ϵ:

$$\left(\frac{\epsilon}{1+\epsilon}\right)^k\left(\frac{\epsilon}{1-\epsilon}\right)^l \leqslant \frac{\epsilon}{1-\epsilon}. \tag{4.3}$$

If ϵ satisfies the conditions of our Lemma, the inequality (4.3) implies (4.2), which proves the statement (4.1). ∎

Remark 2. According to (3.29) in our case $A=\{\ell_x, \ell_y\}$, $B=\{\ell, \ell'\} \subset L$, $\theta_A=(\theta_{\ell_x}-\theta_{\ell_y})$, $\theta_B=(\theta_\ell-\theta'_\ell)$ and $a=1$, $b=e^{\beta\mu_0}$, i.e.,

$$\epsilon = \min\{1/2, 1/(1+e^{-\beta\mu_0})\}. \tag{4.4}$$

Collecting the estimates and identities (3.24), (3.28), (3.29), (4.1), we obtain the lower bound for the two-spin correlation function in the ferrofluid:

$$\langle F_{x,y}\rangle_{\Lambda,\beta,\mu} \geqslant C\epsilon^2\,\frac{(\prod_{j\in L}\int_{S^1} d\phi_j)(\phi_{\ell_x}, \phi_{\ell_y})\, e^{-\beta V^\epsilon(\phi_L)}}{Z_{L,\epsilon}} := C\epsilon^2\langle F_{x,y}\rangle_{V^\epsilon}. \tag{4.5}$$

Therefore, this function is bounded from below by mean of the correlation function of the $O(2)$-ferromagnetic model with interaction:

$$V^{\epsilon}(\phi_L) := -\epsilon^2 \sum_{\{\ell, \ell'\} \subset L} J^0_{\ell\ell'}(\phi_\ell, \phi_{\ell'}). \tag{4.6}$$

Using again the Ginibre inequalities, the correlation function in the right-hand side of (4.5) can be bounded from below by the correlation function of the *nearest-neighbour* ferromagnetic plane-rotator model on the lattice $L \subset \mathbb{Z}^2$, i.e.,

$$\langle F_{x,y} \rangle_{V^{\epsilon}} \geqslant \langle F_{x,y} \rangle_{V^{\epsilon}_{nn}}, \tag{4.7}$$

where

$$V^{\epsilon}_{nn}(\phi_L) := -\epsilon^2 \sum_{\langle \ell, \ell' \rangle \subset L} J^0(\phi_\ell, \phi_{\ell'}), \tag{4.8}$$

and $J^0 := J^0_{\ell\ell'} > 0$ for $|\ell - \ell'| = a$ by condition $(J1)$. We thus arrive at our main result:

Theorem 2. Let two-dimensional $O(2)$-ferrofluid model be defined by the Hamiltonian (2.4) with interactions satisfying the conditions $(u1)$, $(u2)$, $(J1)$, $(J2)$ for some $a - 4R = \delta > 0$. If temperature and chemical potential verify for some $\mu_0 \geqslant 0$ the condition

$$e^{\beta(A_+ + 2\|J\| - \mu + \mu_0)} \leqslant \delta^2, \tag{4.9}$$

then there exists $\beta_0(\delta) > 0$ such that the two-spin correlation function:

$$\langle F_{x,y} \rangle_{\beta,\mu} := \lim_{\Lambda \to \mathbb{R}^2} \langle F_{x,y} \rangle_{\Lambda,\beta,\mu} \tag{4.10}$$

of this model does not decay faster than some inverse power of $|x - y|$ for all $\beta > \beta_0(\delta)$.

Proof. Notice that by conditions of the theorem one has: $J^0 > 0$, see (2.7), and $\epsilon = 1/2$, see (4.4). Therefore, by[14] there is $\beta_0(\delta)$ such that the two-spin correlation function $\lim_{L \to \mathbb{Z}^2} \langle F_{x,y} \rangle_{V^{\epsilon}_{nn}}$ does not decay faster than some inverse power of $|x - y|$ for low temperatures $\beta > \beta_0(\delta)$. Since the condition (4.9) on the temperature and the chemical potential ensures the lower bound (4.5) with $C > 0$, see (3.7), by the estimate (4.7) one gets the same conclusion for the correlation function (4.10). ▮

5. CONCLUDING REMARKS

1. Exponential decay for high temperatures. Notice that by the Ginibre inequality the spin–spin correlation function (3.4) reaches its maximum at the *closed packed* configuration ($\mu = \infty$). Since by standard arguments for lattice systems[13] this function decays exponentially for sufficiently high temperatures $\theta > \theta^* > \theta_0$ even for the closed packed configuration, one gets the same decay everywhere in domain $\{-\infty < \mu < \infty\} \times \{\theta > \theta^*\}$.

2. Algebraic decay for low temperatures. As we see from (4.9) the best bound for the domain of algebraic decay corresponds to the choice $\mu_0 = 0$. For a given $\delta > 0$ By Theorem 2 the Berezinskiĭ–Kosterlitz–Thouless phase is localized for a given $\delta > 0$ by the conditions:

$$\{\theta : \theta < \theta_0(\delta)\} \cap \{\mu : \mu \geqslant A_+ + 2\,\|J\| - 2\theta \ln \delta\} := D_\delta^{\mathrm{BKT}}. \qquad (5.1)$$

Note that by condition ($J1$) the increasing of $\delta > 0$, implies that J^0 (and temperature $\theta_0(\delta)$) will decrease, for a *monotonous decreasing* function $J(t)$, see condition ($J1$). In particular, for a *finite-range* potential $J(t)$ there exist $\delta_{\max}$ such that $J^0 = 0$ for $\delta > \delta_{\max}$. We thus arrive at the (schematic) phase diagram represented on Fig. 1, which corresponds to domain $D^{\mathrm{BKT}} := \bigcup_{\delta > 0} D_\delta^{\mathrm{BKT}}$.

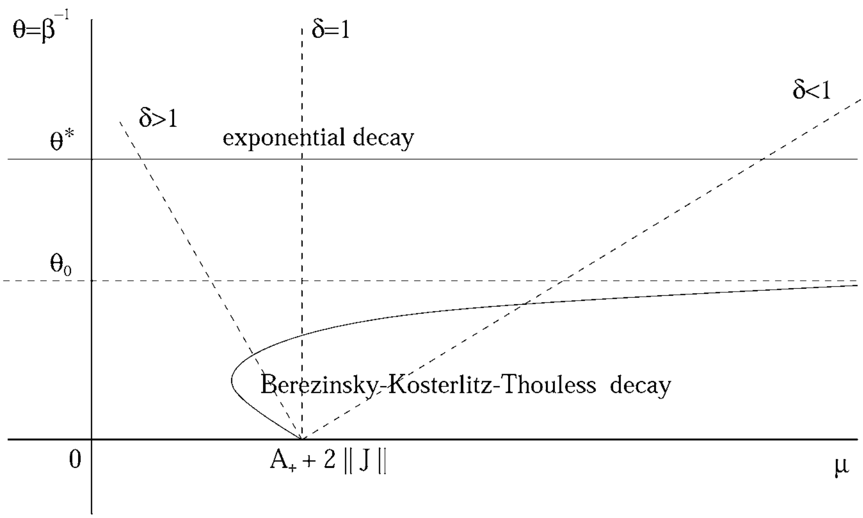

Fig. 1. Schematic phase diagram for two-dimensional ferrofluid of plane rotators.

3. Long-range order for low temperatures. Let the potential $J(t)$ be a long-range one of the form:

$$J(t) = \kappa t^{-(d+\sigma)}, \tag{5.2}$$

for $\kappa > 0$ and $0 < \sigma < d = 2$. Then by construction of the interaction $J^0_{\ell\ell'}$ one gets that

$$J^0_{\ell\ell'} \geqslant \kappa_0(\delta)\, |\ell - \ell'|^{-(d+\sigma)} := J^{\mathrm{LR}}_{\ell\ell'} \tag{5.3}$$

for some $\kappa_0(\delta) > 0$. Denote the potential (4.6) with interaction $J^{\mathrm{LR}}_{\ell\ell'}$ by $V^{\epsilon}_{\mathrm{LR}}$. Then by virtue of (5.3) and the Ginibre inequalities

$$\langle F_{x,y} \rangle_{V^\epsilon} \geqslant \langle F_{x,y} \rangle_{V^\epsilon_{\mathrm{LR}}}. \tag{5.4}$$

Since by refs. 25 and 26 there exists $\beta_0(\kappa_0(\delta))$ such that in the model with interaction $V^{\epsilon}_{\mathrm{LR}}$ one has the order parameter (magnetization) $m_\epsilon(\beta)$:

$$\lim_{|x-y| \to \infty} \lim_{L \to \mathbb{Z}^2} \langle F_{x,y} \rangle_{V^\epsilon_{\mathrm{LR}}} := m^2_\epsilon(\beta) > 0 \tag{5.5}$$

for all $\beta > \beta_0(\kappa_0(\delta))$, by (4.5), (5.4), and (5.5) we conclude the same for our ferrofluid model. Thus, for the long-range ferromagnetic interaction (5.2) the two-dimensional $O(2)$-ferrofluid has in domain $D^{\mathrm{LRO}} := \bigcup_{\delta > 0} D^{\mathrm{LRO}}_\delta$, the *Long-Range Order* parameter $m(\beta, \mu) > 0$:

$$m(\beta, \mu)^2 := \lim_{|x-y| \to \infty} \langle F_{x,y} \rangle_{\beta,\mu} \geqslant C\epsilon^2 m^2_\epsilon(\beta) \tag{5.6}$$

with $\epsilon = 1/2$. Here

$$D^{\mathrm{LRO}}_\delta := \{\theta : \theta < \beta_0^{-1}(\kappa_0(\delta))\} \cap \{\mu : \mu \geqslant A_+ + 2\,\|J\| - 2\theta \ln \delta\}. \tag{5.7}$$

4. Dimensions $d \neq 2$. Notice that by[25, 26] the statement above is also valid for the long-range potential (5.2) when $d = 1$. By the same authors it is known that for low temperatures the *Long-Range Order* parameter (magnetization) is non-null in the model (4.6) even for *short-range* interactions if dimension $d > 2$. Then the estimate (4.5) implies that our ferrofluid model (2.4) has a non-zero magnetization in domain (5.1).

5. Anisotropic rotators. We do not discuss here some straightforward (or less evident) generalizations of our main result to the case of non-plane anisotropic rotators.[27, 28] We return to this question elsewhere.

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